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Simultaneous Approximation and Interpolation with Norm Preservation

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1. In his book, Ref [1], J. L. Walsh gives the following result:

Let T be a compact subset of the complex plane and $f(z)$ a complex-valued function defined on T with the property that for every $\epsilon > 0$ there exists a polynomial $p(z)$ such that $|p(z) - f(z)| < \epsilon$ for all $z \in T$. Then for any positive n, any n distinct points $z_i \in T$ $(1 \leqslant i \leqslant n)$, and any $\epsilon > 0$, there exists a polynomial $q(z)$ such that $|f(z) - q(z)| < \epsilon$ for all $z \in T$ and $q(z_i) = f(z_i)$, $i = 1, 2,..., n$ (i.e., if f can be approximated uniformly by polynomials, it can be approximated so by polynomials which interpolate f at n distinct points).

If, in this theorem, T is chosen to be a compact interval of the real line and $f(x)$ is a continuous real-valued function defined on T, then Walsh's theorem represents a strengthening of the classical Weierstrass approximation theorem. (An additional argument is needed to ensure that the polynomials can be chosen to be real.)

The following theorem, essentially due to W. Wolibner [3] (cf. Ref. [2]), shows that the approximating polynomials of the strengthened Weierstrass theorem can be chosen to satisfy still another condition.

Let $f(x)$ be a continuous real-valued function defined on a nondegenerate compact real interval T. Then for any positive integer n, any n distinct points t_i $(1 \leqslant i \leqslant n)$ belonging to T, and any $\epsilon > 0$, there exists a polynomial $p(t)$ such that $|f(t) - p(t)| < \epsilon$, $t \in T$, $p(t_i) = f(t_i)$ $(1 \leqslant i \leqslant n)$, and

$$\max_{t \in T} |p(t)| = \max_{t \in T} |f(t)|.$$

In Ref. [4], H. Yamabe gave the following abstract version of Walsh's theorem.

Let D be a dense convex subset of a real normed linear space X and let x_i^* $(1 \leqslant i \leqslant n)$ belong to the dual space. Then for every $x \in X$ and every $\epsilon > 0$ there exists $d \in D$ such that $x_i^*(d) = x_i^*(x)$ $(1 \leqslant i \leqslant n)$ and $\| x - d \| < \epsilon$.

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In Ref. [5], I. Singer gave a slightly more general version of Yamabe's theorem:

Let D be a dense convex subset of a real topological vector space X and let x_i^* $(1 \leqslant i \leqslant n)$ belong to the dual space. Then for every $x \in X$ and every neighborhood $N(x)$ of x, there exists $d \in N(x) \cap D$ such that $x_i^*(d) = x_i^*(x)$ $(1 \leqslant i \leqslant n)$.

In Ref. [6], F. Deutsch proved Singer's theorem under the hypothesis that D is a dense *subspace* of the (possibly) complex topological vector space X.

Finally, in Ref. [2], Deutsch and Morris, while generalizing the work of Wolibner, investigated the problem of approximating an arbitrary point of a real normed linear space X by a point d of a dense subspace $D \subset X$, with the side conditions $\| d \| = \| x \|$ and $x_i^*(d) = x_i^*(x)$ $(1 \leqslant i \leqslant n)$, where the x_i^* $(1 \leqslant i \leqslant n)$ belong to the dual space of X. They considered the cases: X a Hilbert space, X a reflexive Banach space, $X = C[T]$, where T is a compact Hausdorff space, and X an L_p space, $1 < p < \infty$. In the present note we prove a theorem which resulted from an attempt to extend the work on this problem by allowing D to be an arbitrary dense convex *subset* of X.

Following Ref. [2], we make the following

DEFINITION. Let D and Y be subsets of a normed linear space X, and let x_i^* $(i = 1, 2,..., n)$ belong to the dual space X^*. The triple $\langle Y, D, \{x_1^*,..., x_n^*\}\rangle$ is said to have property SAIN (simultaneous approximation and interpolation with norm preserved) if, for every $y \in Y$ and every $\epsilon > 0$, there exists $d \in D$ such that $\| d \| = \| y \|$, $x_i^*(d) = x_i^*(y)$ $(i = 1, 2,..., n)$ and $\| y - d \| < \epsilon$.

Let X be a normed linear space and let $x_1^*,..., x_n^*$ be points of X^*. If $x \in X$, we define the interpolation set of x, S_x, relative to $x_1^*,..., x_n^*$, by

$$S_x = \{y \in X \mid x_i^*(y) = x_i^*(x) \ (i = 1,..., n)\}.$$

We call $y \in S_x$ minimal if and only if for each $z \in S_x$, $\| y \| \leqslant \| z \|$. We define also

$$T_x = \{y \in S_x \mid y \text{ is minimal}\}.$$

We can now state our principal result.

THEOREM 1.[1] *Let M be a dense convex subset of the real normed linear space X, and let $x_1^*,..., x_n^* \in X^*$. Then $\langle X, M, \{x_1^*,..., x_n^*\}\rangle$ has property SAIN if and only if $\langle T_x, M, \{x_1^*,..., x_n^*\}\rangle$ does, for each $x \in X$.*

[1] We are indebted to the referee for his comment that Theorem 1 and the following results are also valid in complex spaces. This follows from the fact that Yamabe's theorem continues to hold in complex spaces, as was shown in Ref. [7, pp. 356–357].

Proof. Necessity of the condition is obvious. To show sufficiency, we must prove that, for each $x \in X$ and each $\epsilon > 0$, there exists an $m \in M \cap S_x$, with $\| x \| = \| m \|$ and $\| x - m \| < \epsilon$. To find such an m, we first assume that $x \notin T_x$ and look for $m_1, m_2 \in M \cap S_x$ such that $\| m_1 \| < \| x \| < \| m_2 \|$ and $\| x - m_i \| < \epsilon$ $(i = 1, 2)$; having found such, the convexity of $M \cap S_x$ will be seen to yield the desired m. Since $x \notin T_x$, there exists $y \in S_x$ such that $\| x \| > \| y \|$. The function $g(\lambda) = \| \lambda(2x - y) + (1 - \lambda)y \|$ is convex on $[0, 1]$, and since $g(0) < g(1/2) < g(1)$, it is strictly monotone increasing in a neighborhood of $1/2$. The convexity of S_x implies that

$$\lambda(2x - y) + (1 - \lambda)y \in S_x$$

for $\lambda \in [0, 1]$. The continuity and strict monotonicity of g in a neighborhood of $1/2$ insure the existence of λ_1, λ_2 $(0 \leqslant \lambda_1 < 1/2 < \lambda_2 \leqslant 1)$ such that if

$$z_i = \lambda_i(2x - y) + (1 - \lambda_i)y \qquad (i = 1, 2),$$

then

$$\| z_1 \| < \| x \| < \| z_2 \|$$

and

$$\| x - z_i \| < \epsilon/2 \qquad (i = 1, 2).$$

By Yamabe's theorem we can find $m_i \in M \cap S_x$ such that $\| z_i - m_i \| < \epsilon'/2$ $(i = 1, 2)$, where $\epsilon' = \min(\epsilon, \| z_2 \| - \| x \|, \| x \| - \| z_1 \|)$. Thus, $\| m_1 \| < \| x \| < \| m_2 \|$ and $\| x - m_i \| < \epsilon$ $(i = 1, 2)$. Define the vector-valued function $h(\lambda) = \lambda m_1 + (1 - \lambda) m_2$, $\lambda \in [0, 1]$. The continutiy of the norm insures the existence of $\lambda_3 \in [0, 1]$ such that $\| h(\lambda_3)\| = \| x \|$. Observe that $\| x - h(\lambda_3)\| \leqslant \lambda_3 \| x - m_1 \| + (1 - \lambda_3)\|x - m_2 \| < \epsilon$. Thus, we may may take $m = h(\lambda_3)$.

If we now assume that $x \in T_x$, then, by hypothesis, there exists an $m \in M \cap T_x \subset T_x \subset S_x$ satisfying $\| m - x \| < \epsilon$. We have proved that whether or not $x \in T_x$, we can always find an $m \in M \cap S_x$ such that $\| x - m \| < \epsilon$ and $\| x \| = \| m \|$, i.e., $\langle X, M, \{x_1^*, \ldots, x_n^*\}\rangle$ has property SAIN.

COROLLARY. *Let X be a strictly convex real normed linear space, and let M be a dense convex subset of X. If $x_1^*, \ldots, x_n^* \in X^*$, then $\langle X, M, \{x_1^*, \ldots, x_n^*\}\rangle$ has property SAIN if and only if M contains each minimal point.*

Proof. Because the norm is strictly convex, each T_x consists of at most one point.

In a Hilbert space X with inner product $(\ , \ )$, any $x^* \in X^*$ uniquely determines an $x \in X$ such that for each $y \in X$, $x^*(y) = (x, y)$. We call x the Riesz representation of x^*.

THEOREM 2. *Let M be a dense convex subset of the real Hilbert space X and let $x_1^*,\dots, x_n^*$ be elements of X^*. If $x_1 ,\dots, x_n$ are, respectively, the Riesz representation of $x_1^*,\dots, x_n^*$, then $\langle X, M, \{x_1^*,\dots, x_n^*\}\rangle$ has property SAIN if and only if M contains $\langle x_1 ,\dots, x_n\rangle$, the subspace spanned by $x_1 ,\dots, x_n$.*

Proof. Without loss of generality we may assume that $x_1 ,\dots, x_n$ are orthonormal. Since Hilbert spaces are strictly convex, we need only prove that $\bigcup_{x\in X} T_x$ coincides with $\langle x_1 ,\dots, x_n\rangle$. Each $x \in X$ may be written in the form $x = \sum_{i=1}^n (x, x_i)\, x_i + x^{\perp}$, (where $(x^{\perp}, x_i) = 0$ $(i = 1, 2,\dots, n)$. Further y belongs to S_x if and only if $(x, x_i) = (y, x_i)$ $(i = 1, 2,\dots, n)$; hence, if and only if $y = \sum_{i=1}^n (x, x_i)\, x_i + y^{\perp}$. Since $\| y \|^2 = \sum_{i=1}^n |(x, x_i)|^2 + \| y^{\perp} \|^2$, it follows that $y \in T_x$ if and only if $\| y^{\perp} \| = 0$, i.e., if and only if $y = \sum_{i=1}^n (x, x_i)\, x_i$.

COROLLARY (Deutsch and Morris). *Let M be a dense subspace of the Hilbert Space X and let $x_1^*,\dots, x_n^*$ be elements of X^*. Then*

$$\langle X, M, \{x_1^*,\dots, x_n^*\}\rangle$$

has property SAIN if and only if each x_i^* attains its norm on the intersection of the unit ball with M.

Proof. Because a Hilbert space is strictly convex, each nonzero x_i^* assumes its norm at exactly one point of the unit sphere, viz., $y_i = x_i/\| x_i \|$. Hence, the desired conclusion.

J. D. Stafney [8] has shown the following

THEOREM. *Let $\omega(m)$ be a positive number for $m = 0, 1,\dots,$ such that $\lim_{m\to\infty} \omega(m)^{1/m} = \infty$. Given $f \in C[0, 1]$, with $f(0) = 0$, and a positive number ϵ, there exists a polynomial $P(x) = \sum c_m x^m$ such that $| f(x) - P(x)| < \epsilon$ throughout $[0, 1]$, and $| c_m | < \epsilon \cdot \omega(m)$ for $m = 0, 1,\dots$. ($C[0, 1]$ denotes the linear space of all continuous real-valued functions on $0 \leqslant x \leqslant 1$, with the maximum norm.)*

As a matter of fact, c_0 may be taken to be 0. Let M be the set of all polynomials $\sum_{m\geqslant 1} c_m x^m$ with $| c_m | < \omega(m)$. This set is convex and Stafney's theorem says that M is dense in the subspace $C_0[0, 1]$ of $C[0, 1]$, consisting of the functions which vanish at $x = 0$. Consider the continuous linear functional $I(f) = \int_0^1 f(t)\, dt$. The only minimal element in $\langle C_0[0, 1], M, I\rangle$ is $f \equiv 0$. Therefore $\langle C_0[0, 1], M, I\rangle$ has property SAIN and we can approximate f in $C_0[0, 1]$ by a $P \in M$ which has the same norm as f and is such that $I(f) = I(P)$. It is noteworthy that neither M nor any of its subsets is a dense subspace of $C_0[0, 1]$.

References

1. J. L. Walsh, "Interpolation and Approximation by Rational Functions in the Complex Domain," American Mathematical Society, Providence, R.I., 1935.
2. F. Deutsch and P. D. Morris, On simultaneous approximation and interpolation which preserves the norm, *Bull. Amer. Math. Soc.* **75** (1969), 812–814.
3. W. Wolibner, Sur un polynome d'interpolation, *Colloq. Math.* **2** (1951), 136–137.
4. H. Yamabe, On an extension of the Helly's theorem, *Osaka Math. J.* **2** (1950), 15–17.
5. I. Singer, Remarque sur un théorème d'approximation de H. Yamabe, *Atti. Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* **26** (1959), 33–34.
6. F. Deutsch, Simultaneous interpolation and approximation in topological linear spaces, *SIAM J. Appl. Math.* **14** (1966), 1180–1190.
7. F. Deutsch and P. D. Morris, On simultaneous approximation and interpolation which preserves the norm, *J. Approximation Theory* **2** (1969), 355–373.
8. J. D. Stafney, A permissible restriction on the coefficients in uniform polynomial approximation to $c[0, 1]$, *Duke Math. J.* **34** (1967), 393–396.